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Comments on Chiral *p*-Forms

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Two issues regarding chiral p-forms are addressed. First, we investigate the topological conditions on spacetime under which the action for a nonchiral p-form can be split as the sum of the actions for two chiral p-forms, one of each chirality. When these conditions are not met, we exhibit explicitly the extra topological degrees of freedom and their couplings to the chiral modes. Second, we study the problem of constructing Lorentz-invariant self-couplings of a chiral p-form in the light of the Dirac–Schwinger condition on the energy-momentum tensor commutation relations. We show how the Perry–Schwarz condition follows from the Dirac–Schwinger criterion and point out that consistency of the gravitational coupling is automatic.

1. INTRODUCTION

The talk by M.H. at the workshop on Quantum Gravity in the Southern Cone II, Bariloche, was devoted to the results obtained in refs. 1–3 on the flip of sign in the quantization condition for k-brane dyons (k odd) in $2k + 4$ dimensions. To avoid repetition with what can be found in the literature, the present contribution does not reproduce the actual content of the talk, but deals with the Lagrangian formulation of chiral p-forms. We refer the reader interested in the quantization condition for k-brane dyons to refs. 1–4 for a detailed discussion. See also refs. 5–7 for related information.

Chiral p-forms, i.e., p-forms, the field strength of which is self-dual, can exist in $(2p + 2)$-Minkowski spacetime for any even p. They are notoriously known to suffer from one major difficulty: even though their equations of motion are manifestly Lorentz-invariant, there is no simple (e.g., quadratic in the free case), manifestly Lorentz-invariant Lagrangian that leads to these equations of motion [8].

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Although there is no simple manifestly Lorentz-invariant Lagrangian, there is a simple non-manifestly Lorentz-invariant Lagrangian, which has been given in refs. 9 and 10 and which generalizes the Lagrangian of ref. 11 for chiral bosons. This Lagrangian is linear in the first-order time derivatives of the spatial components of the p-form potential and reads, in the 2-form case that we shall consider for definiteness,

$$S[A_{ij}] = \int dx^0\, d^5 x\, B^{ij} \partial_0 A_{ij} - \int dx^0\, H \qquad (i, j, \ldots = 1, \ldots, 5) \tag{1.1}$$

with

$$B^{ij} = \frac{1}{3!} \epsilon^{ijklm} F_{klm}, \qquad F_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} - \partial_\lambda A_{\nu\mu} \tag{1.2}$$

and

$$H = \int d^5 x\, (N\mathcal{H} + N^k \mathcal{H}_k) \tag{1.3}$$

Here, N and N^k are the standard lapse and shift [12]. The magnetic field B^{ij} is a spatial tensor density of weight one. We are considering from the outset the theory in a gravitational background as in refs. 9 and 10. In the absence of self-interactions, the energy density $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{\sqrt{g}} B^{ij} B_{ij} \tag{1.4}$$

where the spatial indices are lowered and raised with the spatial metric and its five-dimensional inverse, while g is the determinant of g_{ij}. The energy density generates displacements normal to the slices of constant x^0. The momentum density $\mathcal{H}_k$, on the other hand, is purely kinematical and generates tangent displacements. It is explicitly given by

$$\mathcal{H}_k = \tfrac{1}{2} \epsilon_{ijmnk} B^{ij} B^{mn} \tag{1.5}$$

In order to write the action (1.1), it is necessary to assume that spacetime has the product form “time × space.” This will be done throughout in the sequel.

The action (1.1) is manifestly invariant under the gauge transformations

$$\delta_\Lambda A_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i \tag{1.6}$$

since B^{ij} is gauge-invariant and identically transverse ($\partial_i B^{ij} \equiv 0$).[3] In flat space, it is also invariant under Lorentz transformations, but these do not take the usual form [9, 10].

Knowing the action for a chiral form in a gravitational background, one can compute the gravitational anomaly by usual quantum field-theoretic methods [13] and compare the result with calculations based on the nonchiral action supplemented by an appropriate projection [14]. As shown in ref. 13 there is agreement.

The first part of this paper is motivated by this result and aims at understanding better the relationship between the nonchiral action and the chiral ones. We show that when the spatial sections have vanishing Betti numbers b_2 and b_3, the action for a nonchiral form is just the sum of the actions for two uncoupled chiral forms of opposite chiralities. Thus the path integral for a nonchiral 2-form supplemented by a projection to one chiral sector trivially reduces to the path integral for the corresponding chiral modes. This is no longer true for more general topologies. The nonchiral action and the sum of the chiral actions agree on the local degrees of freedom, but treat differently the harmonic components of the 2-form. However, one can easily keep track of the topological "zero mode" difference. This is explicitly done in Section 4, after we have reviewed the necessary background on the dynamics of chiral p-forms. The importance of global features when dealing with chiral forms has been pointed out and stressed in ref. 15, where the problem of modular invariance has been addressed. Recent developments relevant to the six-torus case are given in ref. 16.

In an interesting series of papers [17], a manifestly covariant formulation of chiral p-forms has been developed. This formulation is characterized by the presence of an extra field and an extra gauge invariance. This extra field occurs nonpolynomially in the action, even for free chiral 2-forms. The manifestly covariant formulation has proved useful for many conceptual developments. It has been shown to be equivalent to the non-manifestly covariant treatment of ref. 9 in Minkowski space [18]. To the extent that the analysis of ref. 17 strongly relies on the Poincaré lemma, it is expected to share also similar global features.

The second question analyzed in this paper is that of Lorentz-invariant self-couplings (as well as consistent self-couplings in an external gravitational background) for chiral p-forms. In view of its relevance to the M-theory five-

[3] Since A_{0i} does not occur in the action—even if one replaces $\partial_0 A_{ij}$ by $\partial_0 A_{ij} - \partial_i A_{0j} - \partial_j A_{i0}$ (it drops out because B^{ij} is transverse)—the action is of course invariant under arbitrary shifts of A_{0i}. It is also invariant under arbitrary shifts of any other field that does not appear in the action.

brane, this question has received a lot of attention, both at the level of the equations of motion [19] and at the level of the action [20–24]. We show that this question can be handled by means of the Dirac–Schwinger condition on the commutation relations of the components of the energy-momentum tensor [25, 26]. This condition leads directly to the differential equation obtained in ref. 20 and implies automatically consistency of the gravitational coupling. So, once Lorentz-invariant self-interacting chiral p-form theories have been found, there is no extra work to be carried out to couple them to gravity. The Dirac–Schwinger criterion, which appears to be quite powerful in the present context, has been used recently in ref. 27 in the investigation of Lorentz invariance of manifestly duality-invariant theories in the other even (0 mod 4) spacetime dimensions.

2. DYNAMICS OF CHIRAL 2-FORM

As stated above, we assume that spacetime takes the product form $T \times \Sigma$ where T is the manifold of the time variable (usually a line). Furthermore, we also assume that the spatial sections Σ are either homeomorphic to R^5 (in which case the theory must be supplemented by falloff conditions at infinity that insure the vanishing of the relevant surface terms), or compact. Of course, a spatial coordinate could equivalently play the role of the time variable, as in ref. 20.

We define the exterior form B to be the (time-dependent) spatial 2-form with components $B_{ij}/\sqrt{g}$. The equations of motion that follow from the action are [9, 10]

$$d[N(E - B)] = 0 \tag{2.1}$$

where E is the electric spatial 2-form defined through

$$E_{ij} \equiv \frac{\mathring{A}_{ij} - N^s F_{sij}}{N} \tag{2.2}$$

and where d is the spatial exterior derivative operator. In the case where the second Betti number b_2 of the spatial sections vanishes, this equation implies $N(E - B) = dm$, where m is an arbitrary spatial 1-form. To bring this equation to a more familiar form, one sets $m_i = A_{0i}$. The equations of motion read then

$$F = *F \tag{2.3}$$

where $F_{0ij} = \mathring{A}_{ij} - \partial_i A_{0j} + \partial_j A_{0i}$. This is the standard self-duality condition. Alternatively, one may use the gauge freedom to set $m = 0$, which yields the self-duality condition in the temporal gauge.

To deal with the case where b_2 is not zero, one uses the Hodge decomposition of exterior forms on the spatial sections [28]. Any form—and in particular, any 2-form—can be written as the sum of an exact form, a coexact form, and a harmonic form,

$$A = d\rho + \delta\phi + \sum_A \lambda_A(t)\omega^A \tag{2.4}$$

Here, the codifferential δ acting on a p-form is equal to $\delta = (-1)^{5p} * d*$, while ρ (respectively, ϕ) is a spatial 1-form (respectively, spatial 3-form) and $\{\omega^A\}$ is, on each spatial slice, a basis of harmonic (= closed and coclosed) 2-forms. These satisfy $\partial_i(\omega^{Aij}\sqrt{g}) = 0$, $\partial_{[i}\omega^A_{jk]} = 0$ and are normalized so that $\int d^5x\, \omega^{Aij}\omega^B_{ij}\sqrt{g} = \delta^{AB}$ for each t. The harmonic forms are finite in number and thus the harmonic component of A describes a finite number of global "zero modes." In the simple case where $b_2 = 0$, there are no zero modes. In the case where $b_2 \neq 0$, A may have a nontrivial harmonic part. The equation of motion (2.1) implies in that case

$$N(E - B) = \sum_A k_A(t)\omega^A + dm \tag{2.5}$$

Again, one can absorb the exact part of the right-hand side of (2.5) in a redefinition of E (or set it equal to zero by a gauge transformation), but there is an additional piece which is not determined, namely, the harmonic part. However, this harmonic part turns out to be pure gauge, because the action (1.1) for a chiral 2-form has more gauge invariances than expressed by (1.6). It is actually invariant under addition to A of an arbitrary closed (and not necessarily exact) 2-form,

$$\delta_{\lambda,\epsilon} A = d\lambda + \epsilon_A\omega^A \tag{2.6}$$

This follows because B is coexact (and not just coclosed), and invariant under (2.6).[4] One can thus gauge away the harmonic part of $N(E - B)$ and get again the self-duality condition. Therefore, the action (1.1) leads to the correct self-duality condition, but is a theory in which the zero modes of A are pure gauge (no physical component along the harmonic forms). A similar phenomenon was described in ref. 10 for chiral bosons on a circle. To summarize: for a chiral 2-form, both the exact and the harmonic components (i.e., the closed part of A) are pure gauge and it is the coexact part only that contains the physical degrees of freedom.

For an antichiral 2-form, the action is

$$S[A_{ij}] = -\int dx^0\, d^5x\, B^{ij}\,\partial_0 A_{ij} - \int dx^0\, H \tag{2.7}$$

[4] If the spatial metric depends on t, the 2-form ω^A_{ij} also will be time dependent. The time derivatives $\dot{\omega}^A_{ij}$ are clearly closed so that $\int B^{ij}\dot{\omega}^A_{ij}\, d^5x = 0$.

with $H = \int d^5x\, (N\mathcal{H}' + N^k\mathcal{H}'_k)$. The energy density $\mathcal{H}'$ is the same as for a chiral form, but the momentum density $\mathcal{H}'_k$ differs in the sign. The analysis proceeds exactly as above and one finds this time the antichiral condition

$$E + B = 0 \tag{2.8}$$

An antichiral 2-form described by the action (2.7) has no physical harmonic component.

For later purposes, we shall need the brackets of the gauge-invariant magnetic fields B^{ij}. The orthodox way to proceed is to define conjugate momenta and follow the Dirac method for constrained systems [29]. The chirality condition appears as a mixture of second-class constraints and first-class constraints, the first-class part being related to the gauge invariance of the theory [10]. One may work out the Dirac bracket of the gauge-invariant fields by using the Dirac formula, but one may shortcut the whole procedure and directly read the brackets from the action (1.1), which is already in first-order form. Either way, one finds as Dirac brackets (we consider the chiral case for definiteness, the antichiral one differs in the sign)

$$[B^{ij}(\mathbf{x}), B^{mn}(\mathbf{x}')] = \frac{1}{4}\epsilon^{ijmnk}\delta_{,k}(\mathbf{x} - \mathbf{x}') \tag{2.9}$$

We shall also need the brackets of the energy densities $\mathcal{H}(\mathbf{x})$ at two different space points. A direct calculation using only the form of $\mathcal{H}$ and the brackets (2.9) yields

$$[\mathcal{H}(\mathbf{x}), \mathcal{H}(\mathbf{x}')] = (\mathcal{H}^k(\mathbf{x}) + \mathcal{H}^k(\mathbf{x}'))\delta_{,k}(\mathbf{x} - \mathbf{x}') \tag{2.10}$$

The relation (2.10), derived first on general grounds in refs. 25 and 26, is deeply connected to Lorentz invariance and gravitational coupling and we shall return to it below.

3. ZERO MODES OF A NONCHIRAL FORM

The action for a nonchiral 2-form is

$$S[A_{\mu\nu}] = -\frac{1}{2\cdot 3!}\int d^6x\, \sqrt{-{}^6g}F^{\lambda\mu\nu}F_{\lambda\mu\nu} \tag{3.1}$$

We keep the same notations for the 2-form, even though $A_{\mu\nu}^{\text{here}} \neq A_{\mu\nu}^{\text{before}}$ [see relationship (4.5) below between nonchiral and chiral 2-forms]. It is invariant under the gauge transformations

$$\delta_\Lambda A_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \tag{3.2}$$

which enable one to set A_{0i} equal to zero. The exact part of A_{ij} can then be also gauged away at any given time, *but the harmonic part cannot*. Indeed, the action (3.1) is *not* invariant under shifts of A_{ij} by an arbitrary closed form, only under shifts of A_{ij} by an arbitrary exact form.[5] Thus, the harmonic part of a nonchiral 2-form describes true physical degrees of freedom.

It is easy to see that on a flat background, the harmonic part of A_{ij} behaves like a free particle, i.e., grows linearly with time,

$$A = d\rho + \delta\phi + \sum_A \lambda_A(t)\omega^A \tag{3.3}$$

with

$$\lambda_A(t) = C_A t + D_A \tag{3.4}$$

on-shell. This is because the equation $\partial_\mu(\sqrt{-^6g}F^{\mu\nu\sigma}) = 0$ implies $(d^2\lambda_A/dt^2)\omega^A = \delta(\text{something}) + d(\text{something}')$ and thus $d^2\lambda_A/dt^2 = 0$. Now, if the integration constant C_A is different from zero, the form $A_{\mu\nu}$ cannot be purely chiral or antichiral. Indeed, if it is chiral (say), then, the chirality condition implies

$$C_A\omega^A + d\dot{\rho} + \delta\dot{\phi} = \delta(\text{something}) \tag{3.5}$$

which leads to a contradiction unless $C_A = 0$. Accordingly, if one decomposes the field strength into self-dual part and anti-self-dual part, there is *no* potential for either the self-dual part or for the anti-self-dual part when $C_A \neq 0$, although there is a potential for the sum.

The situation is the same as for a chiral boson φ on a circle. The zero mode $\varphi_0 = at + b$ cannot be written as the sum of single-valued left-movers and right-movers unless $a = 0$, even though the sum is single-valued [$at = (a/2)(t + \sigma) + (a/2)(t - \sigma)$, but $t + \sigma$ or $t - \sigma$ is not single-valued]. Of course, the field strength $F_\mu = \partial_\mu\varphi$ is decomposable into well-defined self-dual and anti-self-dual parts, but these do not derive from a single-valued potential.

We thus see that a nonchiral 2-form contains additional global degrees of freedom besides the local degrees of freedom described by the local chiral actions. It is the presence of the physical zero modes that is responsible for the fact that the sum of a chiral 2-form and an antichiral 2-form is not a nonchiral 2-form on a topologically nontrivial background.

[5] More precisely, the transformation $\delta_\varepsilon A_{ij} = \epsilon_A(t)\omega^A_{ij}$ of the spatial components cannot be supplemented by a transformation of A_{0i} such that $\delta_\varepsilon F_{0ij} = 0$ for arbitrary ϵ's. Indeed, this would require $\dot{\epsilon}_A\omega^A_{ij} + \epsilon_A\dot{\omega}^A_{ij}$ = exact form, which forces ϵ^A to be a solution of the differential equation $\dot{\epsilon}_A + t^B_A\epsilon_B = 0$, where $\dot{\omega}_A = t^A_B\,\omega^B + d(\text{something})$, showing that ϵ^A cannot be an arbitrary function of time. The transformations with ϵ solution to this equation should be regarded as rigid symmetries, not gauge symmetries.

4. DECOMPOSITION OF NONCHIRAL ACTION

In order to compare the action for a nonchiral 2-form with the sum of the chiral and antichiral actions given above, it is convenient to rewrite the nonchiral action in Hamiltonian form. To that end, one follows the Dirac method. One finds

$$S[A_{ij}, A_{0i}, \pi^{ij}] = \int dx^0\, d^5x\, [\pi^{ij}\dot{A}_{ij} - \frac{N}{\sqrt{g}}(\pi^{ij}\pi_{ij} + \frac{1}{4}B^{ij}B_{ij}) - N^k\pi^{ij}F_{kij} - 2A_{0i}\pi^{ij}_{\ ,j}] \tag{4.1}$$

where π^{ij} is the momentum conjugate to A_{ij}. The component A_{0i} appears as a Lagrange multiplier for Gauss' law constraint

$$\partial_i\pi^{ij} = 0 \tag{4.2}$$

One can solve Gauss' law for π^{ij} and eliminate the corresponding multiplier from the action. The general solution of (4.2) is

$$2\pi^{ij} = \frac{1}{2}\epsilon^{ijklm}\partial_k Z_{lm} + \sqrt{2g}\mu_A\omega^{Aij} \tag{4.3}$$

While the 2-form A_{ij} is determined up to an exact form, the 2-form Z_{ij} is determined up to a closed form,

$$\delta_{\Lambda',\chi}Z_{ij} = \partial_i\Lambda'_j - \partial_j\Lambda'_i + \chi_A\omega^A_{ij} \tag{4.4}$$

Using (4.3) and making the change of variables of ref. 2,

$$Z_{ij} = \sqrt{2}(U_{ij} + V_{ij}), \qquad A_{ij} = \sqrt{2}(U_{ij} - V_{ij}) \tag{4.5}$$

one finds, after straightforward algebra

$$\begin{aligned} S[U_{ij}, V_{ij}, \mu_A] = {} & S^{\text{chiral}}[U_{ij}] + S^{\text{antichiral}}[V_{ij}] \\ & + \int d^6x\, \mu_A\sqrt{g}\omega^{Aij}(\dot{U}_{ij} - \dot{V}_{ij}) \\ & - \frac{1}{2}\int d^6x\, N\,\mu_A\omega^A_{ij}\epsilon^{ijklm}\partial_k(U_{lm} + V_{lm}) \\ & - \frac{\sqrt{2}}{2}\int d^6x\, N^k\sqrt{g}\mu_A\omega^{Aij}F_{kij} - \frac{1}{2}\int dt k^{AB}\mu_A\mu_B \end{aligned} \tag{4.6}$$

with

$$k^{AB} = \int d^5x \ \sqrt{g} N \omega^A_{ij} \omega^{Bij} \tag{4.7}$$

The action for the nonchiral form thus splits as the sum of two chiral actions, one of each chirality, plus terms coupling the zero modes μ^A to the chiral components. The action is invariant under the transformations

$$\begin{aligned} \delta_{X,\xi} U_{ij} &= \partial_{[i} X_{j]} + \xi_A(t) \omega^A_{ij} \\ \delta_{X,\xi} V_{ij} &= \partial_{[i} Y_{j]} + \xi_A(t) \omega^A_{ij} \\ \delta_{X,\xi} \mu^A &= 0 \end{aligned} \tag{4.8}$$

with the *same* harmonic component $\xi_A(t)$ for $\delta_{X,\xi} U_{ij}$ and $\delta_{X,\xi} V_{ij}$. Consequently, because of the zero mode coupling, the action (4.6) has fewer gauge invariances than the sum of two chiral actions. The zero mode of the difference $U_{ij} - V_{ij}$ is also gauge invariant. One easily verifies that it is in fact canonically conjugate to μ^A. For a flat metric, the couplings between the local degrees of freedom and the zero modes simplify because the motion is an isometry so that the time derivative of a harmonic form is harmonic. One can disantangle the zero modes from the coexact ones, but this will not be done here.

When $H^2_{DR} \neq 0$, the physical Hilbert space for a nonchiral two-form is bigger than the product of the Hilbert spaces for a chiral two-form and an antichiral one. One must also include the states associated with the harmonic modes,

$$\mathcal{H}^{\text{nonchiral}} = \mathcal{H}^{\text{chiral}} \otimes \mathcal{H}^{\text{antichiral}} \otimes \mathcal{H}^0 \tag{4.9}$$

The truncation to the chiral sector is particularly simple when there are no global, topological zero modes, since it simply amounts then to dropping the uncoupled antichiral degrees of freedom. How to handle the global modes in the general case depends on the context and will not be addressed here.

For issues that depend on the local (high-energy) behavior of the theory, such as anomalies in local symmetries, the topological modes should not be relevant. In the absence of such modes, the change of variables (4.5) can be implemented easily in the path integral and yields

$$\begin{aligned} Z &= \int DA \, D\pi \exp i(S[A, \pi]) \\ &= \int DU \, DV \exp i(S^{\text{chiral}}[U] + S^{\text{antichiral}}[V]) \end{aligned} \tag{4.10}$$

where the measures $DA\,D\pi$ and $DU\,DV$ involve of course the ghost modes and gauge conditions. Note that neither the change of variables (4.5) nor the parametrization (4.3) (when there is no ω^A) involves the metric. Projecting out to the chiral sector by inserting a delta function $\delta(\pi^{ij} - B^{ij})$ of the chirality condition is equivalent to setting the antichiral component V_{ij} to zero, leaving one with the path integral for a chiral 2-form. Thus, implementing the chiral condition by a projection or dealing with the non-manifestly invariant chiral action are clearly equivalent in the absence of harmonic modes.

5. LORENTZ-INVARIANT SELF-COUPLINGS AND SELF-COUPLINGS IN A GRAVITATIONAL BACKGROUND

When one can use the tensor calculus, it is rather easy to construct interactions that preserve Lorentz invariance. These interactions should also preserve the number of (possibly deformed) gauge symmetries (if any), but this aspect is rather immediate for p-form gauge symmetries—although it is less obvious for the extra gauge symmetry of ref. 17.

There is an alternative way to control Lorentz invariance. It is through the commutation relations of the energy-momentum tensor components. Because the energy-momentum tensor is the source of the gravitational field, the method gives at little extra price a complete grasp of the gravitational interactions. As shown by Dirac [25] and Schwinger [26], a sufficient condition for a manifestly rotation- and translation-invariant theory (in space) to be also Lorentz invariant is that its energy density fulfills the commutation relations (2.10). The condition is necessary when one turns to gravitation. The method is more cumbersome than the tensor calculus when one can use the tensor calculus, but has the advantage of being still available even when manifestly invariant methods do not exist.

In the Dirac–Schwinger approach, the question is to find the most general $\mathcal{H}$ fulfilling (2.10). The energy-density $\mathcal{H}$ must be a spatial scalar density in order to fulfill the kinematical commutation relations $[\mathcal{H}(\mathbf{x}), \mathcal{H}_k(\mathbf{x}')] \sim \mathcal{H}(\mathbf{x}')\delta_{,k}(\mathbf{x}-\mathbf{x}')$ and depends on A_{ij} through B_{ij} in order to be gauge invariant. In five dimensions, there are only two independent invariants that can be made out of B_{ij},

$$y_1 = -\tfrac{1}{2}B_{ij}B^{ij}, \qquad y_2 = \tfrac{1}{4}B_{ij}B^{jk}B_{km}B^{mi} \tag{5.1}$$

as can easily be seen by bringing B_{ij} to canonical form by a rotation (only B_{12} and B_{34} nonzero; note that in this local frame the only nonvanishing component of $\mathcal{H}^k$ is $\mathcal{H}^5$). Set

$$\mathcal{H} = f(y_1, y_2), \qquad f_1 = \partial_1 f, \quad f_2 = \partial_2 f \tag{5.2}$$

Then, a calculation following the standard pattern and paralleling the free case calculation yields

$$[\mathcal{H}(\mathbf{x}), \mathcal{H}(\mathbf{x}')] = (\Lambda(\mathbf{x})\mathcal{H}^k(\mathbf{x}) + \Lambda(\mathbf{x}')\mathcal{H}^k(\mathbf{x}'))\delta_{,k}(\mathbf{x} - \mathbf{x}') \tag{5.3}$$

with

$$4\Lambda = f_1^2 + y_1 f_1 f_2 + (\tfrac{1}{2}y_1^2 - y_2)f_2^2 \tag{5.4}$$

Requiring that (2.10) be fulfilled gives

$$f_1^2 + y_1 f_1 f_2 + (\tfrac{1}{2}y_1^2 - y_2)f_2^2 = 4 \tag{5.5}$$

which is precisely the equation (31) of Perry and Schwarz with f replaced by $2f$. The Dirac–Schwinger criterion yields thus directly the Perry–Schwarz equation, whose solutions are investigated in ref. 20.

In the flat-space context ($g_{ij} = \delta_{ij}$, $N = 1$, $N^k = 0$), equation (2.10) guarantees that the interactions are Lorentz invariant and no further work is required [25, 26]. It also guarantees complete consistency in a gravitational background because of locality of $\mathcal{H}$ in the metric g_{ij} [30, 31].

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REFERENCES

[1] S. Deser, A. Gomberoff, M. Henneaux, and C. Teitelboim, *Phys. Lett. B* **400** (1997) 80.
[2] S. Deser, A. Gomberoff, M. Henneaux, and C. Teitelboim, *p*-Brane dyons and electric–magnetic duality, *Nucl. Phys. B*, to appear [hep-th/9712189].
[3] S. Deser, M. Henneaux, and A. Schwimmer, *p*-Brane dyons, θ-terms and dimensional reduction, *Phys. Lett. B*, to appear [hep-th/9803106].
[4] M. S. Bremer, H. Lü, C. N. Pope, and K. S. Stelle, Dirac quantisation conditions and Kaluza–Klein reduction, hep-th/9710244.
[5] B. Julia, Dualities in the classical supergravity limits, hep-th/9805083; E. Cremmer, B. Julia, H. Lü, and C. N. Pope, Dualisation of dualities. I, hep-th/9710119.
[6] L. Andrianopoli, R. D'Auria, and S. Ferrara, U-Duality and central charges in various dimensions revisited, hep-th/9612105.
[7] M. Bertolini, R. Iengo, and C. A. Scucca, Electric and magnetic interaction of dyonic *D*-branes and odd spin structure, hep-th/9801110.

[8] N. Marcus and J. H. Schwarz, *Phys. Lett. B* **115** (1982) 111.
[9] M. Henneaux and C. Teitelboim, *Phys. Lett. B* **206** (1988) 650.
[10] M. Henneaux and C. Teitelboim, Consistent quantum mechanics of chiral p-forms, in *Proceedings Quantum Mechanics of Fundamental Systems 2, Santiago 1987*, Plenum Press, New York (1989).
[11] R. Floreanini and R. Jackiw, *Phys. Rev. Lett.* **59** (1987) 1873.
[12] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* , Freeman, San Francisco (1973), Chapter 21.
[13] F. Bastianelli and P. van Neuwenhuizen, *Phys. Rev. Lett.* **63** (1989) 728.
[14] L. Alvarez-Gaume and E. Witten, *Nucl. Phys. B* **234** (1984) 269.
[15] E. Witten, *J. Geom. Phys.* **22** (1997) 103.
[16] L. Dolan and C. R. Nappi, *A modular invariant partition function for the fivebrane*, hep-th/9806016.
[17] P. Pasti, D. Sorokin, and M. Tonin, *Phys. Lett. B* **352** (1995) 59; *Phys. Rev. D* **52** (1995) 427.
[18] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin, and M. Tonin, *Phys. Lett. B* **408** (1997) 135.
[19] P. S. Howe and E. Sezgin, *Phys. Lett.* **394B** (1997) 62; P. S. Howe, E. Sezgin, and P. C. West, *Phys. Lett.* **399B** (1997) 49.
[20] M. Perry and J. H. Schwarz, *Nucl. Phys. B* **489** (1997) 47.
[21] J. H. Schwarz, *Phys. Lett. B* **395** (1997) 191.
[22] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, *Nucl. Phys. B* **496** (1997) 191.
[23] P. Pasti, D. Sorokin, and M. Tonin, *Phys. Lett. B* **398** (1997) 41; I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin, and M. Tonin, *Phys. Rev. Lett.* **78** (1997) 4332.
[24] M. Cederwall, B. E. W. Nilsson, and P. Sundell, *J. High Energy Phys.* **4** (1998) 7.
[25] P. A. M. Dirac, *Rev. Mod. Phys.* **34** (1962) 592.
[26] J. Schwinger, *Phys. Rev.* **130** (1963) 406, 800.
[27] S. Deser and Ö. Sarioğlu, Hamiltonian electric/magnetic duality and Lorentz invariance, hep-th/9712067.
[28] S. I. Goldberg, *Curvature and Homology*, Dover, New York (1982).
[29] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Academic Press, New York (1964); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton, New Jersey (1992).
[30] C. Teitelboim, *Ann. Phys.* (N.Y.) **79** (1973) 542; The Hamiltonian structure of space-time, in *General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein*, A. Held ed., Plenum Press, New York (1980).
[31] C. Teitelboim, Ph. D. Thesis, Princeton University Press, Princeton, New Jersey (1973).